Signals and systems

Lecture 20

The Laplace Transform Method for analysis of Continuous Time Signals and systems

- Introduction :

* The generalization of the CT Fourier transform is known as the laplace transform.
* Fourier transform represents continuous time signals in terms of complex sinusoidals, i.e. $e^{j\omega t}$.
* laplace transform represents continuous time signals in terms of complex exponentials, i.e. e^{-st}
* laplace transform ($\mathcal{L}T$) can be used to analyze the signals or functions which are not absolutely integrable.
* Laplace transform can be applied to the analysis of unstable systems.
* Laplace transform of the impulse response is called system function or transfer function. ($H(s)$).

- Types of laplace transform:

1- bilateral or two sided transform. (Integration range from $-\infty$ to $+\infty$)

2- Unilateral or one-sided transform (Integration range from 0 to $+\infty$)

- Definition of laplace Transform:

Consider the continuous time signal $x(t)$ it's laplace transform is denoted by $X(s)$ and given as:

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt.$$ where s is complex variable

$$s = \sigma + j\omega$$

σ - real part of s, it is called attenuation constant.

$j\omega$ - imaginary part, it is called complex frequency.

The laplace transform pair $x(t)$ and $X(s)$ is represented as

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

- The Unilateral laplace transform is given as

$$X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt$$ → mainly used for causal signals.

- The inverse laplace transform is given as

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{st}\, ds$$

$x(t) = \mathcal{L}^{-1}\{X(s)\}$

This formula involves complex integration, The inverse $\mathcal{L}T$ can be solved ~~usig~~ using partial fraction expansion.

Relationship between Fourier Transform and Laplace transform.

Fourier Transform: $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$

Fourier transform can be calculated only if $x(t)$ is absolutely integrable, i.e. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

We know that $s = \sigma + j\omega$ → Then

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} dt =$$

$$= \int_{-\infty}^{\infty} \{x(t) \cdot e^{-\sigma t}\} \cdot e^{-j\omega t} dt \Rightarrow$$

* The real exponential convergence factor $e^{-\sigma t}$ enables some of the time function $x(t)$ to converge in the complex s plane.

* We find that, laplace transform of $x(t)$ is basically the fourier transform of $x(t) \cdot e^{-\sigma t}$.

if $s = j\omega$, i.e. $\sigma = 0$,

Thus: $X(s) = X(j\omega)\big|_{s=j\omega}$.

* Fourier transform is a special case of laplace transform.

* Laplace transform is basically fourier transform on imaginary ($j\omega$) axis in the s-plane.

The following notations are used to represent LT and inverse LT

$X(s) = L[x(t)]$

or $x(t) \xleftrightarrow{L} X(s)$

$x(t) = L^{-1}[X(s)]$

$\cancel{X(s)} = L^{-1}\{x(t)\}$

time domain signals → small case letters

s domain → upper case letter

Region of Convergence (ROC)

* The laplace transform of a general signal $x(t)$ is $X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$, in general, the range of values of s for which this integral converges is referred to as the region of convergence (ROC) of the laplace transform.

* $X(s)$ can be converged for certain values of s even if the Fourier transform does not exist, because of the effect of the real exponential $e^{-\sigma t}$.

* from relationship $X(s) = F\{x(t) \cdot e^{-\sigma t}\}$ → Hence →

For laplace transform ⇒ The integral

$X(s) = \int_{-\infty}^{\infty} |x(t) \cdot e^{-\sigma t}| < \infty$ → The Laplace transform exists, if this integral is finite for some values of $\sigma > \sigma_0$ or $Re\{s\} > \sigma_0$.

Examples:

① For the following signal determine the LT

$$x(t) = e^{-at}\cdot u(t).$$

Solution: The given signal $x(t)$ is a causal. The limit of integration is therefor from 0 to ∞. Hence

e^{x} converges if x is negative

$$X(s) = \int_0^\infty x(t)\cdot e^{-st}\,dt = \int_0^\infty e^{-at}\cdot e^{-st}\,dt$$

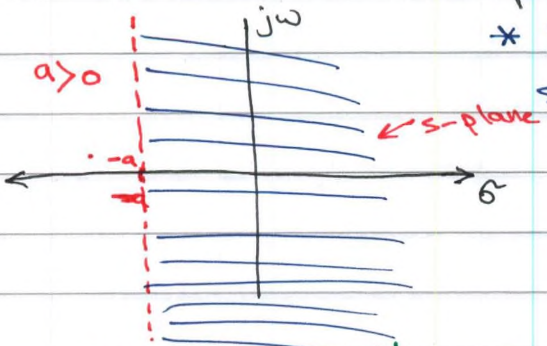
$$= \int_0^\infty e^{-(a+s)t}\,dt = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty = -\frac{1}{s+a}\left[e^{-(s+a)\infty} - e^{-(s+a)\cdot 0}\right]$$

$$X(s) = \frac{1}{s+a}$$

The above integration converges when the upper limit ∞ is applied if $s+a>0$ or $s>-a$, if $s+a<0$, then $e^{-(s+a)\infty}$ does not converge. In such a case LT does not exist.

$$\boxed{e^{-at}\cdot u(t) \xleftrightarrow{L} \frac{1}{s+a},\ Re\{s\}>-a}\ ,\quad \sigma = Re\{s\}$$

For the above example, the RoC is show in figure below:



Roc for $x(t) = e^{-at}\cdot u(t)$.

* we note that fourier transform for this signal exists because jw included in the RoC.

② Consider the following signal

$$x(t) = e^{-at}\cdot u(-t) \qquad \text{Determine the LT.}$$

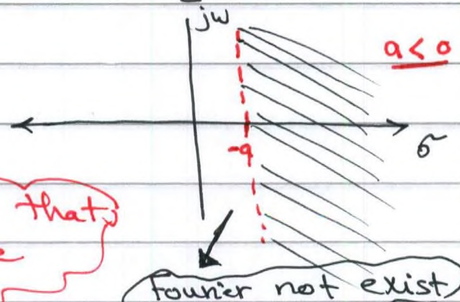
$$X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt = \int_{-\infty}^{\infty} e^{-at}\cdot u(-t)\cdot e^{-st}\,dt = \int_{-\infty}^{0} e^{-at}\cdot e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-(s+a)}\,dt = -\frac{1}{s+a}\left[e^{-(s+a)t}\right]_{-\infty}^{0} = \frac{-1}{s+a}\left[e^{0} - e^{-(s+a)-\infty}\right]$$

$$\boxed{X(s) = \frac{-1}{s+a},\ Re\{s\}<-a}$$

The above integration converges, when

$s+a<0$ or $s<-a$



* Note: The above two examples illustrate that for the same signal $x(t)$, the LT is also same with a ~~sign of~~ change of sign

③

(3) Find the laplace transform for the unit step function:

$$x(t) = u(t)$$

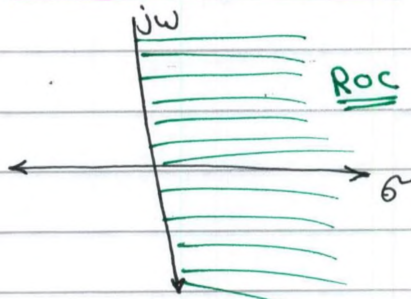
Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt = \int_{-\infty}^{\infty} u(t)\cdot e^{-st}\,dt = \int_{0}^{\infty} e^{-st}\,dt = \frac{1}{-s}\left[e^{-st}\right]_{0}^{\infty}$$

$$= -\frac{1}{s}\left[e^{-s\cdot\infty} - e^{0}\right] \Rightarrow = \frac{1}{s}$$

$$\boxed{e^{-s\cdot\infty} = 0 \Rightarrow s > 0}$$

$$\boxed{u(t) \xleftrightarrow{L} \frac{1}{s}, \quad \sigma > 0}$$



Fourier not exist for $u(t)$.

(4) Find the laplace transform for the unit impulse function:

$$x(t) = \delta(t)$$

$$X(s) = \int_{-\infty}^{\infty} \delta(t)\cdot e^{-st}\,dt = e^{0} = 1, \text{ all } s$$

$$\boxed{\delta(t) \xleftrightarrow{L} 1, \quad \text{Roc all } s}$$

(5) Determine the LT of the following signal, Mark the poles and Roc in the s-plane.

$$x(t) = A\cdot e^{-at}u(t) + B\cdot e^{-bt}\cdot u(-t),$$ where $a>0$, $b>0$, and $|a| > |b|$.

Solution: The given signal $x(t)$ consists of causal and anti-Causal signals and can be written as

$$x(t) = x_1(t) + x_2(t), \text{ where}$$

$$x_1(t) = A\cdot e^{-at}\cdot u(t)$$

$$x_2(t) = B\cdot e^{-bt}\cdot u(-t)$$

$X_1(s)$ is found as follows for a right-sided signal

$$X_1(s) = \int_{0}^{\infty} A\cdot e^{-at}\cdot e^{-st}\,dt = A\cdot\int_{0}^{\infty} e^{-(a+s)t}\,dt = -\frac{A}{s+a}\left[e^{-(s+a)t}\right]_{0}^{\infty}$$

$$x_1(t) \longleftrightarrow X_1(s) = \frac{A}{s+a}, \text{ The Roc is } Re\{s\} > -a$$

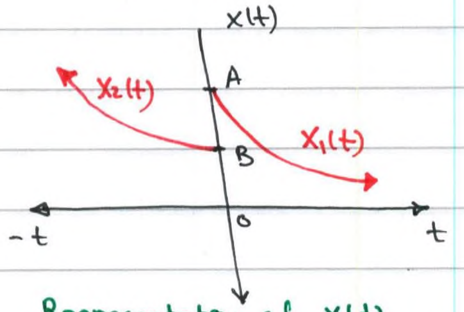
$X_2(s)$ is founded as follows for a left-sided signal

$$X_2(s) = \int_{-\infty}^{0} B\cdot e^{-bt}\cdot e^{-st}\,dt = B\int_{-\infty}^{0} e^{-(s+b)t}\,dt = \frac{-B}{s+b}\left[e^{-(s+b)t}\right]_{-\infty}^{0}$$

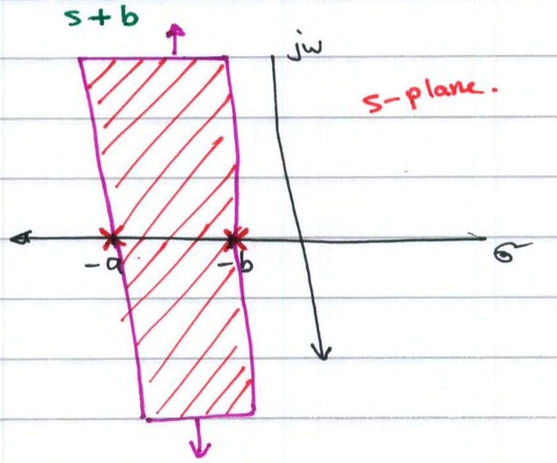
$$= -\frac{B}{s+b}\left[1-0\right] = \frac{-B}{s+b}$$

$$X_2(s) = -\frac{B}{s+b}, \quad \text{Roc: } Re\{s\} < -b$$

$$X(s) = X_1(s) + X_2(s) = \frac{A}{s+a} - \frac{B}{s+b}$$



Representation of x(t)



* The poles are marked using $\underline{\underline{x}}$.

Poles/Zeros Diagram:

Let the Laplace transform of a signal x(t) be a rational function in s, that is

$$X(s) = \frac{B(s)}{A(s)}, \quad \text{for } s \text{ in RoC}$$

where B(s) and A(s) are M-th and N-th order polynomials.

- The M roots of numerator B(s) are called the zeros of the LT.
- The N roots of denominator A(s) are called the poles of the LT
- "x" is used to indicate poles and "o" is used to indicate zeros
- The rational $B(s)/A(s)$ is unbounded for the poles of the LT, Therefore, the poles of $B(s)/A(s)$ lie outside the RoC, the zeros may lie inside or outside the RoC.

⑥ Find the Laplace transform of the signal:

$$x(t) = e^{-at}u(t) + e^{-bt}u(t), \quad a \neq b$$

Solution: $X(s) \; \frac{1}{s+a} + \frac{1}{s+b} = \frac{2s+a+b}{(s+a)(s+b)}, \quad Re\{s\} > \max(-a,-b)$

The poles at $s=-a$ and $s=-b$

The zeros at $s = -(a+b)/2$

There is additional zero when $s \to \infty$.